Reduced Order Models of Refined Yee's Cells

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***Abstract*—This letter introduces a new approach to increasing the accuracy of Finite Difference (*FD*) methods by means of local mesh refinement. The area slightly larger than single Yee's cell is covered by dense mesh and its macromodel is created by the Model Order Reduction (*MOR*) of state equations in the frequency domain. Such macromodels are subsequently used in the Finite Difference Time Domain (*FDTD*) or the Finite Difference Frequency Domain (*FDFD*) analysis of the entire structure. Unlike a popular subgridding technique, the model order reduction approach does not affect the stability or convergence properties of underlying numerical schemes.**

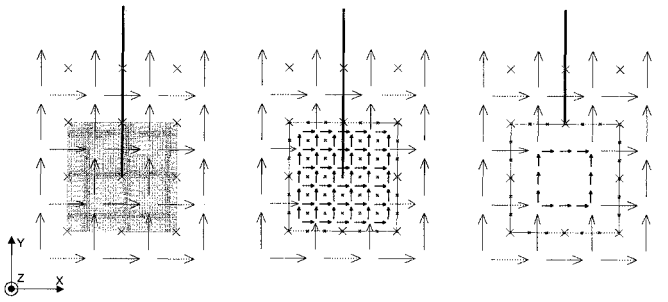
***Index Terms*—FDFD, FDTD, model order reduction.**

I. Introduction

TO IMPROVE the accuracy of the finite-difference formulae in the vicinity of surfaces nonconforming to the grid or small objects which significantly affect the field distribution (e.g., metal wedges, thin wires) local schemes, based on an integral formulation of finite difference equations have been proposed [7], [9]. Local schemes use analytical expressions describing field near a particular object to evaluate the integrals more accurately. An alternative approach, which does not require any analytical formulae, and thus, is suitable for objects of any shape, is to locally condense the mesh. This technique is known as subgridding (*SG*) [10] and while excellent results have been reported with this approach, one evident negative consequence of using a denser mesh is that the norm of the discrete operator increases and this changes the stability or convergence properties of underlying numerical schemes. It is obvious that significant efficiency improvement would be obtained if one could derive a technique similar to subgridding which would not change the norm of the discrete operator. Such a technique would improve the accuracy without increasing the simulation time. In this contribution we show that this can be accomplished by applying the model order reduction technique to the grid equations written for the dense mesh restricted to an area which is small compared to wavelength.

II. Model Order Reduction of a Macrocell

Model order reduction techniques have initially been proposed in computational electrodynamics for finding a compact representation of an electromagnetic problem [1], [2] in the whole computational space. Recently, new applications have been found in which a reduced order model is constructed for a selected subspace. For instance, the analysis of a subvolume containing a small feature using *MOR* in the Finite Elements Method (*FEM*) was carried out in [3]. Similarly, in [5] and [4] subdomain models were developed for the *FDTD* method. Here, we apply the technique described in [4] to create reduced order models of a finely meshed region slightly larger than single Yee's cell.



(a) (b) (c)

Fig. 1. Graphical illustration of the concept of a macrocell and its reduced order model (see the explanations in the text), $\vec{E}_z$ field is represented by crosses, while $\vec{H}_x$ and $\vec{H}_y$ by arrows.

To illustrate the technique, let us consider TM polarized field near a metal wedge situated in *2D* Yee's mesh [Fig. 1(a)] (The metal wedge is merely used as an example, the technique is general and can be used for other problems.) The wedge introduces field singularity which significantly increases an error in finite difference formulae relating the $\vec{H}$ field on the perimeter of the cell containing the wedge to the $\vec{E}_z$ fields in the centers of adjacent cells. To describe more accurately the effect of the singularity on the $\vec{E}_z$ field in these cells, let us consider a rectangular region which covers one entire Yee's cell containing the tip of the wedge and extends to the centers of adjacent cells [Fig. 1(a)]. We shall call this region a macrocell. Assuming uniform meshing, the size of macrocell is $2\Delta x$ by $2\Delta y$, where Δx and Δy is the mesh density in the x and y direction. In order to reduce the error due to singularity, the macrocell is covered by a finer mesh [Fig. 1(b)] and the relation between the $\vec{E}_z$ field on its boundary and the $\vec{H}$ field on the perimeter of the central cell containing the wedge tip is found using Maxwell's grid equations in frequency domain.

To derive this relation, discretized Maxwell's equations on the dense mesh are written in the form of state equations in the s-domain ($s = j\omega$, with ω being the angular frequency)

$$\begin{bmatrix} \mathbf{C_e} & \mathbf{0} \\ \mathbf{0} & \mathbf{C_h} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} + s \begin{bmatrix} \mathbf{0} & \mathbf{D}_\mu \\ \mathbf{D}_\varepsilon & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} \mathbf{f_e} \\ \mathbf{0} \end{bmatrix} \quad (1)$$

where $\mathbf{C_e}$ and $\mathbf{C_h} = -\mathbf{C_e^T}$ are discrete space operators, $\mathbf{D}_\mu$ and $\mathbf{D}_\varepsilon$ are matrices involving parameters such as the discretization step, permeability and permittivity, $\mathbf{e}$, $\mathbf{h}$ are electric and magnetic fields inside a macrocell and $\mathbf{f_e}$ is a vector with ones placed

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at the positions corresponding to an electric field excitation on a macrocell boundary. After field normalization [4] and a simple substitution one obtains

$$\frac{1}{s}\cdot\mathbf{\Gamma}\cdot\mathbf{h}+s\cdot\mathbf{I}\cdot\mathbf{h}=\mathbf{f_e} \tag{2}$$

where $\mathbf{\Gamma}=\mathbf{C_e}\cdot\mathbf{D_\varepsilon^{-1}}\cdot\mathbf{C_e^T}$ is a symmetric, positive semi-definite matrix and $\mathbf{I}$ is the identity matrix. With this equation it is easy to link the excitation ($\vec{E}_z$ fields situated at various locations on macrocell's boundary) with the response [$\vec{H}_t$ field at the perimeter of central, coarse, Yee's cell—see Fig. 1(c)]. This is achieved by transforming (2) into

$$\mathbf{h_M}=\mathbf{L}^T\cdot\left(\frac{1}{s}\cdot\mathbf{\Gamma}+s\cdot\mathbf{I}\right)^{-1}\cdot\mathbf{B}\cdot\mathbf{e_M} \tag{3}$$

where $\mathbf{e_M}$ and $\mathbf{h_M}$ are the electric field at macrocell's boundary and magnetic field being macrocell's response and $\mathbf{L}$, $\mathbf{B}$ are the so-called selector matrices consisting of ones placed at the positions corresponding to the location of excitation or response.

Except for a formal description, the technique described so far is no different from subgridding. However, system (3) can be reduced using the *ENOR* algorithm [6]. The goal of the reduction is to get the low order model of the matrix transfer function relating $\mathbf{e_M}$ with $\mathbf{h_M}$. The *ENOR* algorithm generates a set of $p\cdot q$ orthonormal vectors $\mathbf{V}$, where p is total number of inputs, i.e., size of vector $\mathbf{e_M}$ and q is the model order. This basis is used to project the original system of N equations onto a much smaller system of $p\cdot q$ equations which approximates the desired transfer function with good accuracy. Projecting (3), using the basis $\mathbf{V}$ generated by *ENOR*, one obtains

$$\mathbf{h_M}=\mathbf{L_q^T}\cdot\left(\frac{1}{s}\cdot\mathbf{\Gamma_q}+s\cdot\mathbf{L_q}\right)^{-1}\cdot\mathbf{B_q}\cdot\mathbf{e_M} \tag{4}$$

where $\mathbf{L_q}=\mathbf{V^T L}$, $\mathbf{\Gamma_q}=\mathbf{V^T\Gamma V}$, $\mathbf{I_q}=\mathbf{V^T I V}$, $\mathbf{B_q}=\mathbf{V^T B}$.

The reduced order model of a macrocell gives one the frequency dependent relation between the electric field on the macrocell's boundary and the magnetic field at the chosen location inside a considered space. Since the mesh inside the macrocell is denser than in external Yee's cells, the number of inputs is larger than the number of points in the coarse mesh that are located at the macrocell's boundary. Therefore, reduced order model's excitations at the missing input ports have to be interpolated from the known values of $\vec{E}_z$ fields.

The reduced order model approximates the original system over a limited frequency range. The bandwidth can be increased by increasing the model order q. However, in our case, the reduced order models are constructed for macrocells whose linear dimensions are only twice that of coarse Yee's cell, which means that the frequency dependence is not very significant. As a result, $q=1$ is sufficient to obtain very good accuracy.

A. Using Models in the FDFD and FDTD Algorithms

To apply the reduced order model to the Finite Difference algorithm in the frequency domain, one has to modify Maxwell's equations on the coarse mesh [these equations are analogous to (1)], by replacing the usual finite-difference relations between the $\vec{H}$ fields inside the macrocell and the $\vec{E}_z$ fields at the macrocell boundary with (4). This results in the modification of the $\mathbf{C_e}$ operator defined on the coarse grid. Obviously, since (4) is frequency-dependent, so becomes the modified operator.

TABLE I
NORMS OF THE OPERATOR $\mathbf{\Gamma_q}$ AND $\mathbf{\Gamma}$

k	$\|\|\mathbf{\Gamma_q}\|\|$ (macrocell)		$\|\|\mathbf{\Gamma}\|\|$ - subgridding
	$q=1$	$q=2$	
1	2.73e+09	3.20e+09	3.20e+009
3	1.30e+12	1.66e+25	2.27e+025
9	2.04e+12	6.47e+25	2.30e+026

As for the *FDTD* method, two algorithms for deriving the marching-in-time schemes involving reduced order models have been presented in the literature [4], [5]. Either of them can be used but our numerical tests have shown that the algorithm presented in [5] is faster. The maximum allowable time step in both schemes is inversely proportional to $\sqrt{\|\mathbf{\Gamma_q}\|}$ [8]. For a macrocell and $q=1$ the norm of $\mathbf{\Gamma_q}$ is much smaller than the norm of $\mathbf{\Gamma}$ for external, coarse mesh, even for very large mesh refinement factors. It is quite unlike the norm of $\mathbf{\Gamma}$ for nonreduced refined macrocell. Table I shows examples for a macrocell containing the wedge and nonrefined mesh with $\Delta x=\Delta y=0.5$ mm for different mesh refinement factors (k) and reduced model orders (q). The last column shows the norm of nonreduced operator $\mathbf{\Gamma}$ which determines the stability of the regular subgridding technique. For comparison, the norm of the operator associated with the external mesh, which determines the maximal allowable time step of the *FDTD* algorithm with no refinement, is equal to $2.88e24$, so *FDTD* routine with a macromodel of order $q=1$ can be expected to work stable with the coarse mesh time step even for fine inner meshes. It has to be noted that while the mesh refinement factor k can be large, computational efficiency of our approach decreases as the inner mesh becomes finer.

III. NUMERICAL RESULTS

To verify the method presented in this letter and show its efficiency we have conducted two tests using modified *FD* algorithms. First, an E plane filter (see Fig. 2, the upper-right corner) was analyzed. Six reduced order models of macrocells with the mesh refinement factor $k=9$ where incorporated into the coarse *FDFD* mesh at positions of septa edges (see Fig. 2, the lower-right corner). For comparison, tests of the standard *FDFD* algorithm with coarse and dense mesh were conducted. The coarse mesh discretization step was $\Delta x=\Delta y=5$ mm which gives $N=2000$ variables in a two dimensional *FDFD* algorithm. The fine mesh, with the total number $N=99\,000$ of unknowns, was six times denser than the coarse one. Fig. 2 shows the reflection coefficient for the dominant TE_{10} mode computed with the *FDFD* method together with the results computed with the mode-matching technique used as a reference. The error due to dispersion was estimated and the data corrected so the errors are essentially due to field singularities. As can be seen, the reflection coefficient for the coarse mesh *FDFD* differs considerably from the reference obtained using modal analysis. Differences are still clearly visible even when the mesh is fine. The modified *FDFD* with macrocells allowed us to achieve the

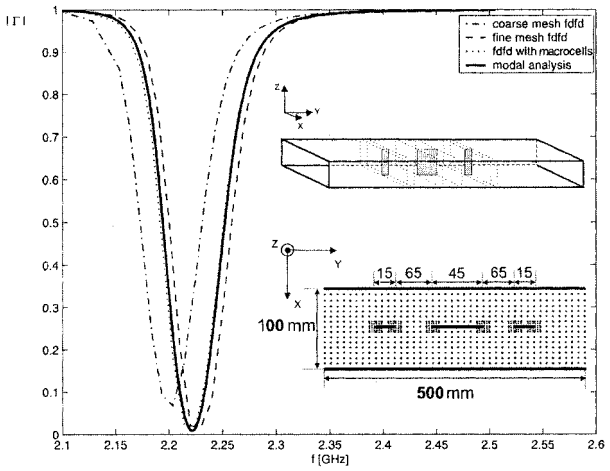

Fig. 2. An E plane filter and its reflection coefficient using the standard and modified *FDFD* algorithm.

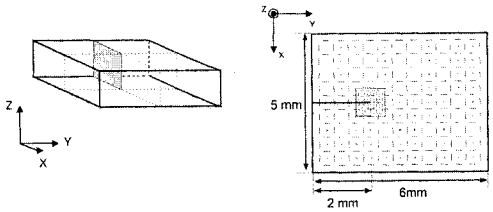

Fig. 3. Structure of a resonator used for testing *FDTD* with a macrocell.

frequency characteristic which is almost indistinguishable from the reference. The CPU time for the modified algorithm was of the same order as that of the standard *FDFD* on the coarse mesh and two orders of magnitude shorter than the computation time for the fine mesh.

To verify the performance of the new algorithm in the *FDTD* method we have calculated the resonant frequency of the first mode in a rectangular resonator containing a metal iris (see Fig. 3). The coarse mesh consisted of 10×12 discretization points ($\Delta x = \Delta y = 0.5$ mm). The whole mesh was subsequently refined by increasing the density by the factor of $k > 1$ in each direction.

Relative errors in the computed resonant frequency with corresponding simulation times for the standard *FDTD* method, *FDTD* with a macrocell (model order $q = 1$) and *FDTD* with subgridding (a scheme similar to [10]) are shown in Table II. The reference value 46.6793 GHz was computed by extrapolating results for progressively refined meshes. In the *FDTD* scheme with a macrocell the same time step, namely that of the standard

TABLE II
COMPARISON OF CPU TIME IN SECONDS AND RELATIVE ERROR IN [%] FOR DIFFERENT TIME DOMAIN ALGORITHMS

	FDTD		$q = 1$		*SG*	
k	time	err	time	err	time	err
1	2.82	-2.60	4.58	-2.60	7.6	-2.60
3	42.2	-0.88	5.18	-0.85	18.7	-0.37
9	1336	-0.30	9.77	-0.27	79.7	0.27
15	6120	-0.18	18.41	-0.16	255.7	0.39

FDTD on the coarse mesh, was used for all refinement factors. Simulations were run for several tens of thousand of iterations and no signs of instability were observed.

IV. CONCLUSIONS

A new technique was presented which allows one to increase the accuracy of the Finite Difference schemes at low time cost. The technique consists in the application of model order reduction to a small region slightly larger than single Yee's cell. The advantage of this approach is that the accuracy increases, while the spectral properties of the underlying operators do not deteriorate.

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